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On the Decomposition Matrices of the Symmetric Groups. II

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In the author's paper [1] the first complete proof was given of the fact that the row of a p-decomposition matrix of S_n corresponding to a p-regular Young diagram contains a 1. Indeed, it was shown that the p-regular diagrams could be ordered in a way that ensured that the relevant 1 was always the last nonzero entry in the row. The purpose of this article is to prove a generalization of these results. In particular, we show that every row of the decomposition matrix contains a 1. The techniques used here are elementary, in the sense that the generalization is obtained by using only properties of Young diagrams.

1. CONSTRUCTION

In order to fix notation, we start by giving some definitions. Throughout, let n be a positive integer and p a prime number.

Consider a fixed origin, and a first axis pointing south and a second axis pointing east. With respect to these axes we have a coordinate system, and we define *vertices* to be elements of $\{(i, j) \mid i \text{ and } j \text{ are positive integers}\}$. A vertex (i, j) is *higher* than (k, l) if $j < l$. Similarly define "*lower* than", "to the *right* of" and "to the *left* of".

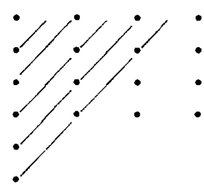
Suppose we have p colors, which we shall call $0, 1,..., p - 1$. Color the vertices by letting (i, j) have the color which is the smallest positive residue of j–i mod p.

A *ladder* is a straight line joining the vertex $(i, 1)$ to the point $\left(1, \frac{i-1}{p-1} + 1\right)$. The vertices through which a ladder passes will be called the *rungs* of the ladder. By construction, every vertex x is a rung of some ladder (called the x-ladder), and

$$\textit{all the rungs of a ladder have the same color.} \tag{1.1}$$

EXAMPLE. $p = 3$.

$$\begin{array}{cccc} 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & & & \\ 1 & & & \end{array}$$



A subset of the rungs of a ladder is a *complete k subset* if it consists of the top k rungs of the ladder.

A *diagram* D for S_n is a subset of size n of the set of vertices having the property that if (i, j) belongs to D then the vertices of $\{(i-1, j), (i, j-1)\}$ belong to D. The vertices which belong to D are called the *nodes* of D.

As usual, a diagram is *p-regular* if no p rows of it have the same length, and otherwise the diagram is *p-singular*.

It is clear that *a diagram D is p-regular if and only if each ladder hits D in a complete k subset, for some k.* (1.2)

Two diagrams D_1 and D_2 *belong to the same block* if and only if they have the same color content (that is, for every i, the number of nodes of D_1 colored i = the number of nodes of D_2 colored i). Nakayama's "Conjecture" then shows that the corresponding ordinary representations belong to the same p-block (see [4, 5.43]).

D_1 is said to *dominate* D_2 (written D_1 doms D_2) if D_1 is not equal to D_2 and for every j $\sum_{i\leqslant j}$ (length of the ith row of D_1) is greater than or equal to $\sum_{i\leqslant j}$ (length of the ith row of D_2). Then "doms" is a partial, but not total, order on the set of diagrams.

CONSTRUCTION. If D is a diagram, construct from D a new diagram D^r as follows. For each ladder 1, if 1 hits D in k nodes, replace these nodes by the complete k subset of 1.

It is easy to check that D^r is, indeed, a diagram. By (1.1) D and D^r belong to the same block, and by (1.2) D^r is p-regular. Also, D^r equals or dominates D.

EXAMPLE. $p = 3$. Looking at

$$\begin{array}{c|ccc} 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & \\ 1 & 2 & 0 & \\ 0 & 1 & & \\ 2 & 0 & & \\ 1 & & & \\ 0 & & & \end{array}$$

we see that if $D = [2, 1^6]$, then $D^r = [4, 3, 1]$. This example shows that there can be a p-regular diagram E ($= [3^2, 1^2]$) such that D^r doms E doms D.

For each diagram D there is an ordinary irreducible representation of S_n. We shall abuse notation, and use D as the symbol for this representation. It is shown in [1] that if D is p-regular then D, when reduced mod p, has a modular irreducible constituent $\varphi(D)$ with multiplicity 1, such that $\varphi(D)$ is a constituent of no D' which dominates D. Theorem 5 of [1] shows that

if D is p-singular, all the irreducible modular constituents of D belong to $\{\varphi(D') \mid D'$ dominates $D\}$ and (1.3)

if D is p-regular, $\varphi(D)$ is a constituent of D with multiplicity 1, *and all the other constituents of D belong to $\{\varphi(D') \mid D'$ dominates $D\}$.* (1.4)

Our main objective is to prove

THEOREM A. *If D is any diagram, then $\varphi(D^r)$ is a constituent of D with multiplicity* 1, *and all the other constituents of D belong to $\{\varphi(D') \mid D'$ dominates $D^r\}$.*

An immediate corollary is

THEOREM B. *Every row of the decomposition matrix of S_n contains a* 1.

For the rest of this paper, if E is a diagram and D is a p-regular diagram we shall write $E = \lambda\varphi(D) + \cdots$, when we mean that E has a modular constituent $\varphi(D)$ with multiplicity λ, and all its other constituents belong to $\{\varphi(D') \mid D'$ dominates $D\}$.

In order to prove Theorem A, we shall assume inductively that

if E is a diagram for S_{n-1}, then $E = \varphi(E^r) + \cdots$ (1.5)

2. Shadow Diagrams

If D is a diagram for S_n and $D\backslash\{x\}$ is a diagram for S_{n-1}, we shall call x a *removable* node of D, and write $D - x$ for $D\backslash\{x\}$. The node x is *regular-removable* if $D - x$ is p-regular diagram.

The node x of D is called a *shadow node* if x is regular removable and no node higher than or equal to x can be raised, retaining its color (that is, if y is a removable node of D at least as high as x, and z is a vertex higher than y which can be added to $D - y$ to give a diagram, then z has a different color from y). Then $D - x$ will be called a *shadow* of D.

EXAMPLE. $p = 5$. Looking at

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 & \\ 3 & 4 & 0 & & & \\ 2 & 1 & & & & \end{array}$$

we see that the nodes at the ends of the first three rows of [6, 5, 3, 2] are shadow nodes, and all the removable nodes of [6, 5, 3, 1] are shadow nodes.

The reason for this nomenclature will become clear when we come to Theorem C. The point is that when a p-regular D is restricted to S_{n-1}, the restriction of $\varphi(D)$ contains at least $\varphi(D')$ for D' a shadow of D.

We shall now show that only p-regular diagrams have a shadow.

Suppose that D is p-singular and x is a regular-removable node of D. Let y be the rung next above x in the x-ladder. Since D is p-singular and $D - x$ is p-regular, y is not a node of D, but a node can be added to D at y to give a diagram. Therefore x can be raised to y retaining its color. Thus, x is not a shadow node.

Now suppose that D is p-regular. Consider the longest ladder 1 which hits D. There are no nodes of D to the right of this ladder, so all the rungs of 1 in D are removable. In particular, the lowest rung x in D is regular-removable. The only nodes higher than or equal to x which are removable are rungs of 1, and so have the same color as x. The only vertices higher than x where a node can be added to D are immediately below a rung, or at the top right hand corner of D. None of these vertices have the same color as x. Hence x is a shadow node. The x which is constructed from a p-regular D in this way will be called the *first shadow node* of D, and $D - x$ is the *first shadow* of D. We have shown that

D has a shadow if and only if D is p-regular. (2.1)

Suppose that D_1 and D_2 are different diagrams in the same block, and x is a shadow node of D_1 and y is a shadow node of D_2. If $D_1 - x = D_2 - y$, then x and y have the same color. Also we may assume that y is higher than x, since $D_1 \neq D_2$. Thus we can raise x to y retaining its color, a contradiction. Therefore

D_1 and D_2 are different p-regular diagrams in the same block implies that no shadow of D_1 is a shadow of D_2. (2.2)

Now let D be an arbitrary diagram S_n. Let $X = \{x_i \mid 1 \leqslant i \leqslant a\}$ be the set of removable nodes of D^r. Let $Y = \{y_i \mid 1 \leqslant i \leqslant b\}$ be the set of regular-removable nodes of D^r. Let $Z = \{z_i \mid 1 \leqslant i \leqslant c\}$ be the set of removable nodes of D. Put an equivalence relation $\sim$ on X by $x_i \sim x_j$ if and only if x_i and x_j are in the same ladder. Then Y provides $\sim$-class representatives. Let $\lambda(y_i)$ be the size of the $\sim$-class containing y_i.

The diagrams $D^r - x_i$ are totally ordered by doms, so we may assume that
$D^r - y_b$ doms $D_r{}^r - y_{b-1}$ doms $\cdots$ doms $D^r - y_1$. (2.3)

Then y_1 is the first shadow node of D^r.

The y_1-ladder hits D^r in $\lambda(y_1)$ nodes and so hits D in $\lambda(y_1)$ nodes. All ladders longer than the y_1-ladder miss D^r and so miss D. The rungs of the y_1-ladder in D are therefore removable nodes of D, and we may take them to be z_i for $1 \leqslant i \leqslant \lambda(y_1)$.

Then, for $1 \leqslant i \leqslant \lambda(y_1)$, $(D - z_i)^r = D^r - y_1$. It is clear that for $i > \lambda(y_1)$, $(D - z_i)^r = D^r - y_j$ for some $j > 1$.

(Note, however, that not every $D^r - y_j$ need turn up as a $(D - z_i)^r$. For instance, if $p = 3$, $D = [1^3]$ has one removable node, but $D^r = [2, 1]$ has two regular-removable nodes.)

Equation (1.5) now gives

$$\begin{aligned} D - z_i &= \varphi(D^r - y_1) + \cdots \text{ if } 1 \leqslant i \leqslant \lambda(y_1) \\ &= \varphi(D^r - y_j) + \cdots \text{ for some } j > 1, \text{ if } i > \lambda(y_1). \end{aligned} \tag{2.4}$$

Let $\downarrow$ denote the process of restricting to S_{n-1}. So $D\downarrow = \sum(D - z_i)$, and (2.3) and (2.4) give

$$D\downarrow = \lambda(y_1)\varphi(D^r - y_1) + \cdots. \tag{2.5}$$

Since $D^{rr} = D^r$, we also have

$$D^r\downarrow = \lambda(y_1)\varphi(D^r - y_1) + \cdots. \tag{2.6}$$

Now, if $x_i \sim y_j$ then $(D^r - x_i)^r = D^r - y_j$, so we have the further result

$$D^r\downarrow \supseteq \lambda(y_j)\varphi(D^r - y_j), \tag{2.7}$$

where this is defined to mean that the restriction of D^r has $\varphi(D^r - y_j)$ as a constituent with multiplicity *at least* $\lambda(y_j)$.

Using these results, we next prove:

THEOREM C. *If D is p-regular (so that $D = D^r$), and y_j is a shadow node of D, then $\varphi(D)\downarrow \supseteq \lambda(y_j)\varphi(D - y_j)$, where $\lambda(y_j)$ is defined as above. If y_1 is the first shadow node of D, then $\varphi(D)\downarrow = \lambda(y_1)\varphi(D - y_1) + \cdots$*

Proof. In view of (2.6) and (2.7), it is sufficient to prove that there is no D' satisfying: $D' \neq D$ and $\varphi(D') \subseteq D$ and $\varphi(D')\downarrow \supseteq \varphi(D - y_j)$. If there were such a D', then $D'\downarrow \supseteq \varphi(D - y_j)$, and (1.4) shows that

(i) D' dominates D, and

(ii) D' has a removable node w such that $D - y_j$ dominates or equals $D' - w$.

It is easy to show that these conditions imply that y_j is lower than w and that we may obtain D' from D as follows. Increase some row of D by 1, decrease a subsequent row by 1, and carry on alternately, finishing by decreasing a row which occurs no later than y_j. Consider the color of the first node which is added to D during this process. Since D' and D are in the same block, this must be the same as the color of some lower node which is removed from D. This contradicts the definition of y_j as a shadow node.

In order to prove Theorem A we return to the general case, where D is an arbitrary diagram. Theorem C gives

$$\varphi(D^r)\downarrow = \lambda(y_1)\varphi(D^r - y_1) + \cdots. \tag{2.8}$$

Suppose that E is a p-regular diagram such that $\varphi(E) \subseteq D$. Then E is in the same block as D and D^r. (2.9)

Suppose that w is the first shadow node of E. Then $D\downarrow \supseteq \varphi(E)\downarrow = \lambda(w)\varphi(E - w) + \cdots$ by Theorem C. (2.5) shows that $E - w$ equals or dominates $D^r - y_1$.

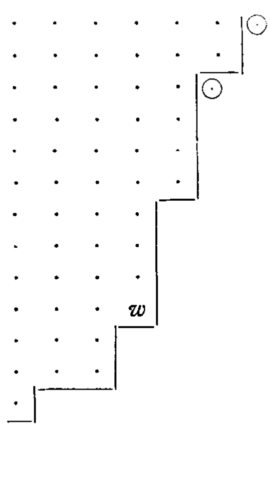
Since $D\downarrow \supseteq \varphi(D^r - y_1)$, there is some such E, say E_1, such that $\varphi(E_1)\downarrow \supseteq \varphi(D^r - y_1)$. Then $D^r - y_1$ equals or dominates the first shadow of E_1. Therefore $D^r - y_1$ equals the first shadow of E_1. (2.2) and (2.9) now give that $D^r = E_1$. Thus $\varphi(D^r)$ is a constituent of D, and its multiplicity must be 1, by comparing (2.5) and (2.8). This proves the first part of Theorem A.

Suppose next that $\varphi(E) \subseteq D$, but $E \neq D^r$. We have to show that E doms D^r in this case.

Since $\varphi(E)\downarrow \not\supseteq \varphi(D^r - y_1)$, $E - w$ must dominate $D^r - y_1$.

Let B be the set of nodes of E not lower than w. Because $D^r - y_1$ is p-regular and $E - w$ dominates $D^r - y_1$, the set of nodes of $D^r - y_1$ not lower than w are contained in B. If y_1 belongs to B or is at least as low as w, then E doms D^r. We show that the other possibilities lead to a contradiction.

EXAMPLE. $p = 5$. $E = [6^2, 5^4, 4^4, 3^2, 1]$. Then $B = [6^2, 5^4, 4^4]$.



Since D^r is p-regular, the remaining possibilities are that y_1 is immediately below the highest rung of the w-ladder, or is to the right of the top right-hand node of E (in the example, at one of the circled vertices). In this case, w and y_1 have different colors. Furthermore, $E - w$ cannot equal or dominate $D^r - y_j$ for $j > 1$. Because $\varphi(E - w)$ is a constituent of $D\downarrow$, (2.4) shows that $\varphi(E - w)$ is a constituent of $D - z_i$ for some i between 1 and $\lambda(y_1)$. Therefore $E - w$ is in the same block as $D - z_i$, which is in the same block as $(D - z_i)^r = D^r - y_1$. The fact that w and y_1 have different colors now contradicts (2.9). This completes the proof of Theorem A.

3. Applications

Recall that the decomposition matrix of S_n has rows labelled by diagrams, and columns labelled $\varphi(D)$, as D runs over p-regular diagrams. We illustrate the techniques developed in this article by giving a quick proof of a theorem of Peel on hook diagrams, including in the theorem a statement of how the columns concerned should be labelled.

A hook diagram of S_n is one of the form $H_j = [j, 1^{n-j}]$.

If $p = 2$, $H_j{}^r$ is of the form $[n - m, m]$, so Theorem A shows that all the constituents of H_j have the form $\varphi[n - l, l]$. These modular irreducibles are all known, from the theorem in [2].

From now on, we shall assume p is odd, and $n = ap + b$, with $0 \leqslant b \leqslant p - 1$.

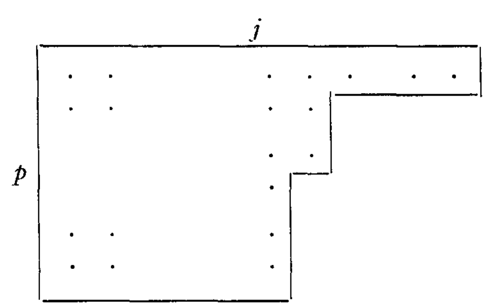
If $b = 0$, then

$$H_a{}^r = H_{a+1}^r = [a + 1, a^{p-2}, a - 1]. \tag{3.1}$$

By considering the longest ladder which hits a hook diagram, one readily proves that apart from the single exception (3.1),

$$H_i{}^r = H_j{}^r \text{ implies that } H_i = H_j. \tag{3.2}$$

If $j \geqslant a + 2$, the shape of $H_j{}^r$ in general is



Hence, if $n > j \geqslant a + 2$, $H_j{}^r$ has at least two removable nodes, and the two highest of these are shadow nodes, provided that $b \neq 0$. (3.3)

THEOREM (Peel [3]). *Suppose that p is odd and $n = ap + b$ with $0 \leqslant b \leqslant p - 1$. Then*

(i) *If $b \neq 0$, $H_j = \varphi(H_j{}^r)$ and so is irreducible. If $i \neq j$, then $\varphi(H_i{}^r) \neq \varphi(H_j{}^r)$.*

(ii) *If $b = 0$, then part of the decomposition matrix is (with zeros omitted)*

$$\begin{array}{rcl|cccccc}
H_1 & = & [1^n] & 1 & & & & & \\
H_2 & = & [2, 1^{n-2}] & 1 & 1 & & & & \\
H_3 & = & [3, 1^{n-3}] & & 1 & 1 & & & \\
 & \vdots & & & & & & & \\
H_{n-1} & = & [n-1, 1] & & & & & 1 & 1 \\
H_n & = & [n] & & & & & & 1
\end{array}$$

The label of the jth column is $\varphi(H_j{}^r)$ for $j < a$, and $\varphi(H^r_{j+1})$ for $j \geqslant a$.

Proof. The result is certainly true when $a = 0$, so we may assume that $a \geqslant 1$ and that the result is true for S_{n-1}. Let $K_j = [j, 1^{n-j-1}]$, defining $K_j = 0$ if $j < 1$ or $j > n - 1$. Then $H_j\!\downarrow = K_j + K_{j-1}$.

Case 1. $b = 1$.

Assume that $n > j \geqslant a + 2$. Then (3.3) and theorem C show that $\varphi(H_j{}^r)\!\downarrow \supseteq \varphi(K^r_{j-1}) + \varphi(K_j{}^r)$. But, by induction, $H_j\!\downarrow = \varphi(K^r_{j-1}) + 2\varphi(K_j{}^r) + \varphi(K^r_{j+1})$.

If $H_j \neq \varphi(H_j{}^r)$, then some other constituent of H_j, say $\varphi(D)$, has the property that D is obtained from either $K_j{}^r$ or K^r_{j+1} by adding a node at the end of the first row (by Theorem C). Then D has the same color content as H_{j+1} or H_{j+2}. Since $n \equiv 1 \pmod p$, this contradicts D and H_j being in the same block. Thus $H_j = \varphi(H_j{}^r)$.

Since H_n is certainly irreducible, we have shown that $H_{a+2}, \ldots, H_n$ are all irreducible. Now, $n \geqslant 3a + 1$ since p is odd, so this gives at least half the hook diagrams. Therefore, by considering the conjugate, H_{n-j+1}, of H_j we see that all the hook diagrams are irreducible. No two are isomorphic, by (3.2).

Case 2. $b \neq 1$.

By Theorem A and induction,

either $H_j = \varphi(H_j{}^r)$ and $\varphi(H_j{}^r)\!\downarrow = \varphi(K^r_{j-1}) + \varphi(K_j{}^r)$

or $H_j = \varphi(D_1) + \varphi(D_2)$ with $\varphi(D_1)\!\downarrow = \varphi(K^r_{j-1})$ and $\varphi(D_2)\!\downarrow = \varphi(K_j{}^r)$, where just one of D_1 and D_2 equals $H_j{}^r$. (3.4)

If $b \neq 0$ and $n > j \geqslant a + 2$, then (3.3) and Theorem C show that $\varphi(H_j^r)\downarrow$ has at least two constituents. Therefore, $H_j = \varphi(H_j^r)$ in this case. The fact that all the hook representations are irreducible again follows by conjugating.

If $b = 0$, then (3.1) and Theorem A show that $\varphi(H_{a+1}^r)$ is a common constituent of H_a and H_{a+1}. From (3.4), we must have that $H_{a+1} = \varphi(H_{a+1}^r) + \varphi(D_2)$, with $\varphi(D_2)\downarrow = \varphi(K_{a+1}^r)$. By Theorem C, D_2 has K_{a+1}^r as a shadow. But H_{a+2}^r also has this as a shadow, and is in the same block as H_{a+1}. Therefore, by (2.2), $D_2 = H_{a+2}^r$. We have now shown that $\varphi(H_{a+2}^r)$ is a common constituent of H_{a+1} and H_{a+2}. Continuing this process we deduce that result (ii) of the theorem correctly describes the constituents of H_j for $j \geqslant a + 1$. Bearing in mind that always $\varphi(H_j^r)$ is a constituent of H_j, the rest of result (ii) now follows by conjugating. This completes the proof of Peel's Theorem.

We now have methods available for obtaining a number of the modular irreducibles for an arbitrary symmetric group. The theorem in [2] gives the part of the decomposition matrix whose rows have labels of the form $[n - m, m]$, and so gives all the modular irreducibles of the form $\varphi[n - m, m]$. Hence we also know the part of the decomposition matrix corresponding to the conjugate diagrams $[2^m, 1^{n-2m}]$. Using Theorem A, we can usually sort out what the column labels for this part should be. Next we can apply Peel's Theorem, if $p \neq 2$, and get even more irreducibles.

EXAMPLE. $p = 3$ and $n = 7$.

	$\varphi[7]$	$\varphi[6,1]$	$\varphi[5,2]$	$\varphi[4,3]$	$\varphi[3^2,1]$	$\varphi[3,2,1^2]$	$\varphi[3,2^2]$	$\varphi[4,2,1]$	$\varphi[5,1^2]$
$[7]$	1								
$[6, 1]$		1							
$[5, 2]$	1		1						
$[4, 3]$			1	1					
$[1^7]$				1					
$[2, 1^5]$					1				
$[2^2, 1^3]$				1		1			
$[2^3, 1]$	1					1			
$[3, 1^4]$							1		
$[4, 1^3]$								1	
$[5, 1^2]$									1

In this example, all of the modular irreducibles turn up in the way described. It is clear that once all the modular irreducibles of S_n are known, the full decomposition matrix can be calculated algorithmically.

It has been much simpler to find $\varphi[4, 2, 1]$ by seeing that $[4, 1^3]$ is irreducible than it would have been to calculate the constituents of $[4, 2, 1]$ (which, in fact, has three constituents besides $\varphi[4, 2, 1]$). Therefore, when tackling the problem of working out the modular irreducibles of successive symmetric groups, one should not necessarily concentrate exclusively on p-regular diagrams. Theorem A will be the key result when considering p-singular diagrams.

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